Scaling transformation of random walk distributions in a lattice

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We use a decimation procedure in order to obtain the dynamical renormalization group transformation (RGT) properties of random walk distribution in a $1+1$ lattice. We obtain an equation similar to the Chapman-Kolmogorov equation. First we show the existence of invariants through the RGT. We also show the existence of functions which are semi-invariants through the RGT. Second, we show as well that the distribution $R_q(x)=[1+b(q-1)x^2]^{1/(1-q)}$ $(q>1)$, which is an exact solution of a nonlinear Fokker-Planck equation, is a semi-invariant for RGT. We obtain the map $q'=f(q)$ from the RGT and we show that this map has two fixed points: $q=1$, attractor, and $q=2$, repellor, which are the Gaussian and the Lorentzian, respectively. We show the connections between these result and the Levy flights.

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The Chapman-Kolmogorov equation is the simplest description of a Markovian process and it is a key concept of statistical mechanics. On the other hand, scaling methods are essential in circumstance where a system is scaling invariant or act as if it was so. As a general rule, whenever a characteristic control length diverges, critical effects occur whose treatment requires renormalization group (RG) methods [1–4]. In this way, scaling on linear lattices becomes a powerful and elegant tool, since these systems have recursive hierarchical geometry.

The main objective of this article is to analyze the evolution of the random walk noise distribution (RWD) in $1+1$ lattice. We obtain the surprising result that the evolution is similar to a Chapman-Kolmogorov equation.

The equation of the motion for the displacement ξ_l at the site l is

$$-4\Omega\xi_l=\xi_{l-1}-2\xi_l+\xi_{l+1}+g_l, \tag{1}$$

where $\Omega=\omega^p(p=1$ diffusion, $p=2$ elastic waves), ω is the frequency in appropriate units, and g_l is a force due to RWD of noise. The renormalization group transformation (RGT) for the equation of motion of a linear lattice, decimation, is well known [4]. The new idea here is to introduce the noise at the site l and to find out its evolution.

We then eliminate the sites $l-1$ and $l+1$ to get

$$-4\Omega'\xi_l=\xi_{l-2}-2\xi_l+\xi_{l+2}+g_l'(t), \tag{2}$$

where $\Omega'=4\Omega(1-\Omega)$ and $g_l'=g_{l+1}+2g_l+g_{l-1}-4\Omega g_l$. For successive iterations we have

$$\Omega_n=4\Omega_{n-1}(1-\Omega_{n-1}), \tag{3}$$

n being the order of iteration. Equation (3) produces a chaotic map in the region $0<\Omega<1$, with density $P(\Omega)\propto[\Omega(1-\Omega)]^{-1/2}$. This distribution has very high probability for numbers close to 0 and 1 [5]. Both gives sums or subtractions of two consecutive distributions, i.e., $g_l\pm g_{l-1}$. Additions or subtractions of random numbers brings to the same result. In any case, the hydrodynamical limit $\Omega\to0$, corresponds to the relevant excitations. For acoustic modes of wavelength λ, $\Omega\approx(\lambda^{-1}a)^2\ll10^{-10}$. Consequently, we may drop the term Ωg_l. Thus the noise transformation becomes

$$g_l'=g_{l+1}+2g_l+g_{l-1}. \tag{4}$$

Therefore each RGT will be the sum of two consecutive sums. For example, consider first a discrete noise $g_j=\Theta m$, with Θ as the noise intensity and $m=\pm1/2$. The initial probability W_0 has the value $W_0(m)=1/2$, while for the first sum the results will be $W_1(\pm1)=1/4$; $W_1(0)=1/2$. The interval has been doubled and the probability has been modified. After n iterations (sums) the probability may be written in the recursive form

$$W_n(k)=\sum_j W_{n-1}(j)W_{n-1}(k-j),\quad n\geqslant1. \tag{5}$$

Notice that $W_n(k)$ has a binomial-type distribution.

In order to obtain the density of probability, or the density distribution, of noise at the iteration $n, P_n(x)$, we first localize x in discrete intervals k of width Δx. We define now $W_n(k)$ as the probability of order n of finding x in a given interval k. Now, in the continuous limit $\Delta x\to0$, we obtain

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$$P_n(x)=\int_{\lambda_1}^{\lambda_2}P_{n-1}(x')P_{n-1}(x-x')dx', \quad n\geqslant 1. \quad (6)$$

Here $|x|<L_n$, $|x'|<L_{n-1}$, and $L_n=2L_{n-1}$. From this condition we get $\lambda_1=(-L_n-x+|x|)/2$, $\lambda_2=(L_n-x-|x|)/2$.

Equation (6) is similar to the Chapman-Kolmogorov equation if we consider the evolution in n as the time evolution. However, our equation is iterative, and the function of order $n,P_n(x)$, for low n may be very different from that of order $n-1$. In this aspect, the left hand side of Eq. (6) may have RWD which are formally different from the right hand side. For large n we expect a RWD which will not change any more. In this aspect $n\rightarrow\infty$ is similar to $t\rightarrow\infty$ in a kinetic equation.

Since Eq. (6) brings us back to the generalized central limit theorem (GCLT), we may ask: what is new? First, in the GCLT each step consists in adding a new random variable x_{N+1} to the sum $(x_1+x_2+\cdots+x_N)/N$, while in the RGT the number of random variables grows exponentially $(N=4^n)$ as iterations succeed. Second, and most fundamental, we are not adding the variables. It is the collective motion which is doing that, indeed, it is the same as Eq. (3) where a given starting frequency generates all the phonon spectrum. Thus if we apply a symmetric and discrete noise of intensity $\pm\Theta$ [as in Eq. (5)] we end up with a Gaussian distribution. Finally, this is a direct connection between RG methods and the Chapmann-Kolmogorov equation.

The recurrence relation defined by Eq. (6) has very important properties. For an even starting distribution, all the subsequent distributions will be even; the distributions will be decreasing functions of $|x|$, with vanishing values for $|x|\geqslant L_n$. The convolution (6) gives

$$\tilde{P}_{n+1}(k)=\tilde{P}_n(k)^2, \quad (7)$$

where $\tilde{P}_n(k)$ is the Fourier transform

$$\tilde{P}_n(k)=\int_{-\infty}^{+\infty}e^{-ikx}P_n(x)dx. \quad (8)$$

Equations (7) and (8) show that if $P_n(x)$ is a normalized function, $\tilde{P}_n(0)=1,P_{n+1}(x)$ is normalized as well.

We shall denote invariants of the renormalization group transformation (IRGT) the functions which keep their form under two consecutive transformations given by Eq. (6). Formally, the IRGT may be read as

$$P_n(x)=bP_{n-2}(bx), \quad (9)$$

in such way that the lattice spacing a transforms as $a'=2a$, and the noise linewidth as $c'=c/b$.

By using direct integration we show that the Gaussian $\exp[-(x/c)^2]$, the Lorentzian $(c^2+x^2)^{-1}$ and the delta function $\delta(x/c)$ are IRGT. Moreover, the set of Lévy functions [6]

$$L(\mu,x)=\frac{1}{2\pi}\int e^{ikx}e^{-(ck)^\mu}dk \quad (10)$$

are IRGT. This can be directly obtained from Eqs. (7) and (9) with $b=2^{-2/\mu}$. The Gaussian $(\mu=2)$, the Lorentzian $(\mu=1)$, and the delta function $(\mu=0)$ are particular cases of Eq. (10). The parameters c scale the same as a only for the Gaussians. That is, Gaussians are commensurate with the lattice. For fractional $\mu,0<\mu<2,L(\mu,x)$ is used in the study of fractal diffusion [7]. Recently Chaves [8] found out a fractal diffusion equation to describe Lévy flights. As a consequence, he predicted the violation of space-inversion symmetry, which was confirmed experimentally [9] with $\mu\approx 1.3$.

Consider now the sequences of functions $P_n(q,x)$. Here q is a continuous parameter $1\leqslant q\leqslant 2$, and x is unbound: $|x|<\infty$. For this sequence we know the function

$$P_0(q,x)=R_q(x), \quad (11)$$

where

$$R_q(x)=A_q[1+\beta(q-1)x^2]^{1/(1-q)}. \quad (12)$$

For large x and $q\neq 1$ Eq. (12) behaves as a power law. It decays more slowly than Gaussians $(q=1)$ and are more appropriate to the study of critical behavior [10]. This RWD has been applied meaningfully to the study of several phenomena such as turbulence [11], anomalous random walk [12], linear response [13], and to scaling properties of multifractal attractors [14]. This distribution is as well the exact solution of a nonlinear Fokker Planck equation [15]. It is not difficult to see that for every μ and c exist a q and b which makes Eq. (12) close to Eq. (10), i.e., curve (12) is curve (10) plus a perturbation.

Using Eq. (12) in to Eq. (6) we obtain a new function $P_1(q,x)$, which in general do not have the same form $R_q(x)$. This is not a surprise since Tsallis recipe is for non-Markovian systems which do not satisfy the Chapman-Kolmogorov equation. However, $P_1(q,x)$ is very close to $R_q(x)$ if the parameters are adjusted to become new parameters q', A_q', and β'. Thus we can say

$$P_1(q,x)\approx R_{q'}(x), \quad (13)$$

or successively $P_n(q,x)=P_{n-1}(q',x)$ until we reach Eq. (13). We shall call this RGT semi-invariant (SRGT). The parameters q, A_q, and β_q transform as

$$q'=f(q), \quad (14a)$$

$$\beta'=\beta b(q), \quad (14b)$$

$$A_{q'}=\frac{A_q^2}{\sqrt{\beta}}i(q). \quad (14c)$$

We suggest a procedure for obtaining f, b, and i (FBI transformation) by making the functions and all derivatives up to the fifth order to agree at the origin. From those we obtain

$$f(q)=q-\frac{4q(q-1)(q-2)}{3q^2-16q+5}, \quad (15a)$$

$$b(q)=\frac{5-q}{4(1+q)}, \quad (15b)$$

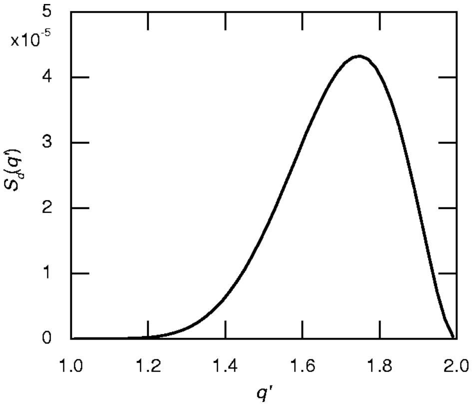

FIG. 1. We plot the standard deviation S_d as a function of q. For every value of q,S_d measure the ''distance'' between the exact $P_1(q,x)$ and the analytical $R_{q'}(x)$ obtained approximately through the RGT. We see that error is meaningless ($S_d<5\times10^{-5}$).

$$i(q)=\frac{4^{2\alpha+1}\pi\Gamma(4\alpha-1)\sqrt{\alpha}}{\Gamma(2\alpha)^2},\quad \alpha=\frac{1}{q-1}. \tag{15c}$$

We shall notice that this result is exact only for $q=1$, Gaussian, and $q=2$, Lorentzian. Consequently *f, b*, and *i* are approximate functions. Before we proceed to analyze the implication of the FBI transformation we shall discuss the error related to the approximation. We define the relative standard deviation

$$S_d(q)\equiv\frac{1}{A}\int |R_{q'}(x)-P_1(q,x)|^2dx, \tag{16}$$

where $A=|\int R_{q'}(x)dx|^2=1$.

In Fig. 1 we plot $S_d(q)$ as a function of q. It shows a maximum around $q\approx1.7$, $S_d(1.7)\approx4\times10^{-5}$. Consequently, even the maximum error is meaningless. We see as well that S_d drops to zero very rapidly as we approach the fixed points. Consequently, for most of the practical purpose we can take Eq. (13) as an equality.

In Fig. 2 we plot $P_1(q,x)$ and $R_{q'}(x)$ for various q' $=f(q)$. The full curve is from the renormalization approach, while the points are from the numerical integral $P_1(q,x)$. Again we see a remarkable agreement.

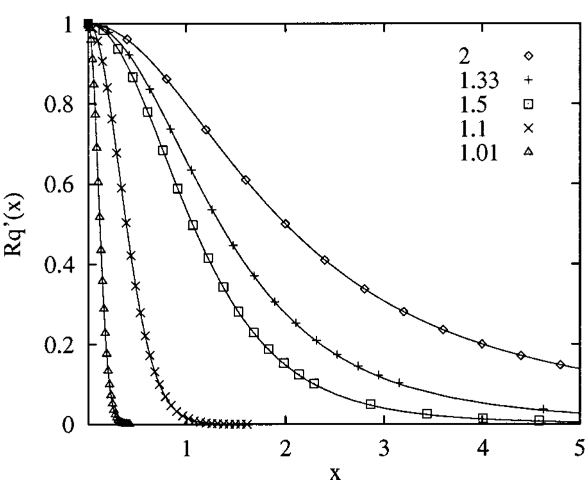

FIG. 2. Density of probability for several values of q. The continuous curve is obtained by the approximated analytical renormalization group transformation, while the points are from exact numerical calculations.

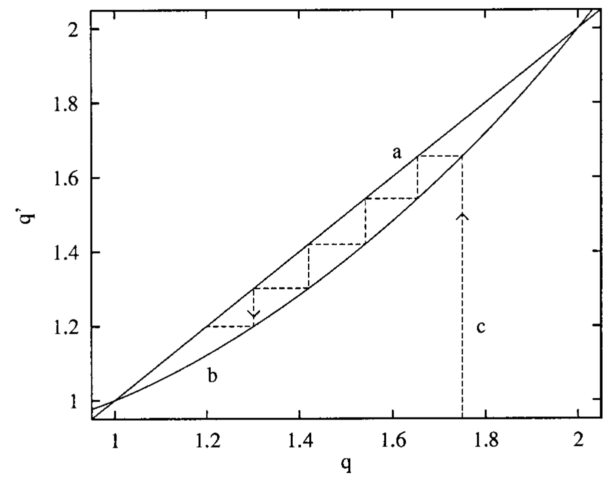

FIG. 3. The return map. We plot q' as function of previous q in the iterative process. In (a) we show the line $q'=q$. In (b) we show the curve $q'=f(q)$ with the fixed points $q^\star=f(q^\star)$ at $q_1^\star=1$ and $q_2^\star=2$. The derivative of the curve shows that only the Gaussian $q_1^\star=1$ is an attractor. In (c) we show a trajectory going from a arbitrary $q\neq q^\star$, approaching successively the stable fixed point.

We return now to the analysis of the Eq. (15). They are the RGT for the Eq. (12) thought the decimation process. Equation (15c) is just the requirement that $R_q(x)$ is normalized, we shall discuss only $f(q)$ and $b(q)$. First Eq. (15a), the main one, gives the map $q'=f(q)$, i.e., how successive q' may be obtained from the previous q.

In Fig. 3 we plot q' as a function of q. The straight line, Fig. 3(a), is the set of the fixed points $q'=q$, while Fig. 3(b) is the map $q'=f(q)$. The map shows two fixed points $q^\star=f(q^\star)$ at $q_1^\star=1$ and $q_2^\star=2$. We shall notice that the requirement for stability $|df(q^\star)/dq|<1$ is fulfilled only for $q^\star=1$. Consequently only the Gaussian is a stable fixed point (attractor). Figure 3(c) shows the trajectory obtained from the returned map. A consequence of the FBI transformation is to reduce the freedom of q. We start with any $q\neq q^\star$ and we get a discrete sequence of transformation towards $q\rightarrow q_1^\star=1$.

Equation (15b) shows another important result. That is how the length scale is modified by our RGT. For the RWD give by Eq. (12), $\langle x^2\rangle\propto1/\beta$. It is well know [16] that $\langle x^2\rangle\propto t^{2/D_f}$, where D_f is the fractal dimension of the random walk. The evolution could be stated as

$$\frac{\langle x^2\rangle_{n+1}}{\langle x^2\rangle_n}=\frac{\beta}{\beta'}=2^{2/D_f} \tag{17}$$

or

$$D_f=\mu=-\frac{\ln(4)}{\ln[b(q)]}. \tag{18}$$

Equation (18) establish a connection between the Lévy flights and the power law (12). Again, D_f is a approximated function being exact only for the fixed points. Note that the ratio (17) holds even if $\langle x^2\rangle$ diverges.

The fractal dimension is a decreasing function of q. For $q=1$, $D_f=2$ and $\langle x^2\rangle \propto t$, while for $q=2$, $D_f=1$, and $\langle x^2\rangle \propto t^2$. For $1\leqslant q\leqslant 2$ we get $2\leqslant D_f\leqslant 1$. This is the same range found for electron diffusion in disordered lattice [17].

We have seen that the Lorentzians and Gaussians are fixed points of both sequences $L(\mu,x)$ and $P_n(q,x)$. This may have some connections with the fact that Gaussians and Lorentzians have integer dimensions. So, there is a commensurability between the scaling of the lattice parameter and the scaling of the linewidth of those curves.

In conclusion, we start with a decimation process for the RWD which brings us to an iterative equation which is similar to the Chapman-Kolmogorov equation. We show the existence of some RWD which are invariants of the equation. We show as well the existence of SRGT, which may be of practical use in the study of RWD governed by power law. These find many applications in physics. In particular, in the last years reasonable amount of work has been done in granular material and surface growth [18] where inelastic collisions take place and the hypothesis of molecular chaos breaks down. For those we expect the results discussed here will be important.

Scaling concepts may enrich the study of nonlinear phenomena [19], in particular, these concerning nonlinear stability, such as fractures [20]. In general, a nucleation (see Ref. [21] and references therein) may occur when fluctuations grow beyond a nonreturn point. The inclusion of noise may shade some light in this important problem. A more detailed discussion will need to include non-Markovians effects. The shape of Eq. (6) will be drastically modified and the possibility of obtaining a stable solution other than Gaussians may be possible. However, at the present state of art, it remains wishful thinking.

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